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Grading refinements in the contractions of Lie algebras and their invariants

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Abstract. All mutually inequivalent toroidal gradings of the Lie algebra $sl(3, \mathbb{C})$ are constructed. Each of them provides several different graded contractions to other eight-dimensional Lie algebras. These can be either continuous, or discrete. The continuous ones are related to generalized Wigner–Inonu contractions. The behaviour of $sl(3, \mathbb{C})$ Casimir operators under the graded contractions is studied and the Casimir and generalized Casimir operators of the contracted Lie algebras are presented.

1. Introduction

Let us consider a physical problem described by a system of equations, be they algebraic, differential, finite difference, integral, or some combination of the above. A crucial feature of the problem is the symmetry group G of the system, i.e. the group of transformations that takes solutions into solutions.

It is always of interest to study relations arising between problems corresponding to different, but related groups G.

One such type of relation between different Lie groups is mutual inclusion: $G_0 \subset G$. Systems invariant under a group G are related to those invariant under a subgroup $G_0 \subset G$ via symmetry breaking.

A different type of relation between Lie algebras (and the corresponding Lie groups) is provided by Lie algebra contractions and deformations. A given Lie algebra L of dimension n is embedded into a family of Lie algebras depending on parameters. All algebras in the family have the same dimension n, but they can belong to different isomorphy classes.

Lie algebra contractions were introduced in a more specific manner by Inonu and Wigner [1], further studied e.g. by Saletan [2] and reviewed e.g. by Gilmore [3].

Wigner–Inonu contractions can be viewed as singular changes of bases, starting from some chosen basis in a given Lie algebra. Indeed, consider a basis $\{e_1, \ldots, e_n\}$ of a Lie algebra L. Introduce a new basis

$$\begin{aligned} f_i &= U_{ik}(\varepsilon_1, \ldots, \varepsilon_p)e_k \\ U_{ik}(1, \ldots, 1) &= \delta_{ik} \end{aligned} \tag{1.1}$$

where U_{ik} is some matrix depending on the parameters ε_i. Let U be nonsingular for $\varepsilon_i \neq 0$, $|\varepsilon_i| < \infty$, but singular when one or more of the parameters go to zero. For $\varepsilon_i \neq 0$ the new

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basis will satisfy commutation relations with structure constants $\tilde{C}_{ikl}$ transformed from the original ones C_{ikl}:

$$\tilde{C}_{ikl} = U_{ia}(\varepsilon)U_{kb}(\varepsilon)C_{abd}U_{dl}^{-1}(\varepsilon). \tag{1.2}$$

For $\varepsilon_i \to 0$, formula (1.2) no longer holds and we obtain a new algebra, the contracted one, in general not isomorphic to L.

Typical examples, bringing out the physical meaning of contractions, are contractions from Poincaré groups $P(n, 1)$ to Galilei groups $G(n)$ in the singular limit $1/c \to \infty$, where c is the speed of light. Another example is the contraction of the de Sitter groups $O(3, 2)$, or $O(4, 1)$ to the Poincaré, or Euclidean groups [4, 5] when the 'radius of the universe' R satisfies $1/R \to 0$.

A different approach has been investigated more recently, namely that of 'graded contractions of Lie algebras' [6–10]. A grading by a finite Abelian group (most often a cyclic group) of automorphisms of L decomposes the Lie algebra into a sum of grading subspaces

$$L = L_0 \dotplus L_1 \dotplus \cdots \dotplus L_M, [L_i, L_k] \subseteq L_{i+k(\mathrm{mod}\, M)} \tag{1.3}$$

where M is the order of the grading.

Instead of modifying the basis we modify the commutation relations in a manner that respects the grading

$$[L_i, L_k] \subseteq \varepsilon_{ik} L_{i+k(\mathrm{mod}\, M)} \tag{1.4}$$

i.e.

$$[x, y]_\varepsilon \equiv \varepsilon_{ik}[x, y] \subseteq L_{i+k} \qquad x \in L_i, y \in L_k.$$

The parameters ε_{ik} do not depend on the choice of x and y, only on the grading subspaces involved (i.e. on i and k).

In order for the new deformed commutation relations to define a Lie algebra, that is to satisfy the antisymmetry condition and the Jacobi relations, the parameters ε_{ik} must satisfy

$$\varepsilon_{ik} = \varepsilon_{ki} \tag{1.5}$$

and

$$\varepsilon_{ij}\varepsilon_{i+j,k} = \varepsilon_{ik}\varepsilon_{i+k,j} = \varepsilon_{jk}\varepsilon_{j+k,i}. \tag{1.6}$$

Once a Lie algebra and a grading are given, a systematic study of graded contractions amounts to solving equations (1.5), (1.6) and then taking all limits $\varepsilon_{ik} \to 0$, compatible with these relations. The system (1.6) has two kinds of solutions. The first are 'continuous' ones: the chosen contraction coefficients ε_{ik} can go to zero continuously, without ever violating the Jacobi relations (1.6). For 'discrete' graded contractions the Jacobi relations hold only after the limit is taken: one algebra changes into another one discontinuously. The continuous graded contractions are closely related to Wigner–Inonu contractions [11].

A recent article [6] was devoted to graded contractions of the Lie algebra $sl(3, \mathbb{C})$. One particular grading was introduced, namely a $\mathbb{Z}_7$ toroidal grading, the finest grading that can be induced by a discrete subgroup of the Cartan subgroup of the corresponding group ($SL(3, \mathbb{C})$ in the case under consideration). This grading is equivalent to the root decomposition of the Lie algebra $L \sim sl(3, \mathbb{C})$.

The problem of classifying and obtaining all graded contractions, corresponding to the finest grading, turned out to be a complex one, even for a relatively low-dimensional simple Lie algebra, such as $sl(3, \mathbb{C})$. Indeed 32 different isomorphy classes of contracted Lie algebras were obtained [6].

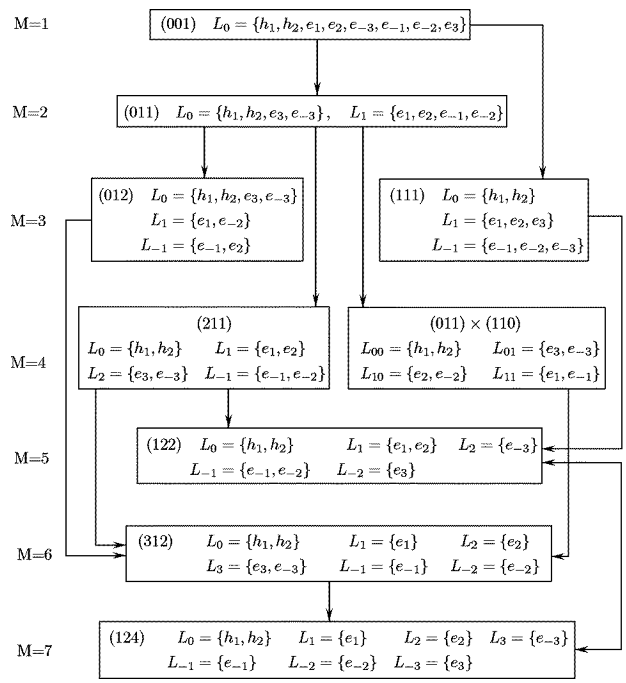

Figure 1. Hierarchy of toroidal gradings of $SL(3, \mathbb{C})$. The levels are determined by the order M of the grading group; each grading subspace L_j is specified by an explicit choice of its generators; the conjugacy class of element generating the grading group is also shown.

In this article we first determine the hierarchy of non-equivalent $sl(3, \mathbb{C})$ gradings (figure 1) by the elements of a maximal torus of the group $SL(3, \mathbb{C})$ (toroidal gradings). Then we consider the contractions preserving individual gradings.

The main idea of our approach is based on the following fact which becomes obvious from the description of our method below. Consider Γ_1 and Γ_2, any pair of $sl(3, \mathbb{C})$ gradings on figure 1 linked by an edge of the graph. Suppose T, is the coarser of the two. Then the sets S_1 and S_2 of nonisomorphic Lie algebras arising by the contractions preserving Γ_1 and Γ_2 respectively, are related by $S_1 \subset S_2$.

Thus having determined all the Lie algebras in S, one needs to find only those in $S_2 \backslash S_1$, when Γ_1 is replaced by Γ_2. The result is not only a considerable economy of effort because the same Lie algebra is not found several times, but more importantly we associate with each contracted Lie algebra the coarsest grading(s) of $sl(3, \mathbb{C})$ which give rise to this Lie algebra.

In section 2 we introduce the gradings of $sl(3, \mathbb{C})$ used in this article. They are generated by an Abelian grading group T_0, a finite group of order M with $2 \leqslant M \leqslant 7$. The actual contractions are performed in section 3, starting from the two coarsest ones. One of them is obtained for $M = 2$, the other for one of the two possible distinct $M = 3$ gradings. All other gradings are refinements of these two. Section 4 is devoted to the invariants of the co-adjoint representation of the considered Lie algebras, or equivalently to the Casimir and generalized Casimir opertors. More specifically, for continuous contractions we show how the two $sl(3, \mathbb{C})$ Casimir operators behave under contraction and we calculate all the corresponding invariants of the contracted Lie algebras.

2. Hierarchy of toroidal gradings

A toroidal grading [12, 13] of a semisimple Lie algebra L is a decomposition of L into the direct sum of linear subspaces, that are eigenspaces of a chosen subgroup T_0 of the maximal torus T of the corresponding Lie group G. Since all choices of the maximal torus of a semisimple Lie algebra L over the field $\mathbb{C}$ of complex numbers are equivalent, we assume that a torus is *a priori* chosen and fixed. Two toroidal gradings of L are considered equivalent, if they can be transformed into each other by an inner automorphism of L.

For a given grading of L the choice of the grading subgroup is not unique. A natural choice is to take this subgroup to be 'as small as possible'.

For all but one toroidal grading of $sl(3,\mathbb{C})$ it suffices to choose the grading group T_0 as a cyclic group of rather low order, $M \leqslant 7$. The desired grading decomposition is obtained as an eigenspace decomposition of L under the action gLg^{-1} of a single element $g \in T$ that generates the cyclic group T_0. In the one remaining case, a $Z_2 \times Z_2$ grading, two elements are needed.

There are practical reasons for considering finite subgroups of the torus as grading groups, rather than continuous ones. First of all, elements of finite order in the torus are well known [10]. Secondly, their action on the Lie algebra and on any of its representations is easily described.

Let us now turn to the task in hand, namely to determine all inequivalent toroidal gradings of the Lie algebra $sl(3,\mathbb{C})$. It is convenient to carry out our computations in the three-dimensional irreducible representation of $sl(3,\mathbb{C})$, the defining representation. Without loss of generality we can choose the maximal torus T to be realized by diagonal matrices $g \in \mathbb{C}^{3\times 3}$ satistying $\det g = 1$. An element $g \in T$ acting on L as an element of order M, i.e. satisfying

$$g^M X g^{-M} = X \tag{2.1}$$

for all $X \in L$, is parametrized by a set of three non-negative mutually prime integers

$$s = [s_0, s_1, s_2] \qquad s_0, s_1, s_2 \in \mathbb{Z}^{\geqslant 0},\ \ gcd\{s_0, s_1, s_2\} = 1 \tag{2.2}$$

such that

$$M = s_0 + s_1 + s_2. \tag{2.3}$$

Explicitly we have [15]

$$g = \text{diag}\{e^{2\pi i(2s_1+s_2)/M}, e^{2\pi i(-s_1+s_2)/M}, e^{2\pi i(-s_1-2s_2)/M}\} \tag{2.4}$$

with M as in equation (2.3)

The non-negative integers s_0, s_1 and s_2 can be visualized as being attached to the nodes of the extended (affine) Dynkin diagram of A_2. A permutation of the labels s_0, s_1 and s_2 only changes the ordering of $sl(3,\mathbb{C})$ roots. Hence, only one ordering need be considered; permutations will give equivalent gradings.

Different choices of s_0, s_1, and s_2 can give equivalent gradings for another reason. This is related to the finiteness of the dimension of L: for a sufficiently high order M of the element g some of the grading subspaces of L will by necessity be empty. Let us now obtain all toroidal gradings of $sl(3,\mathbb{C})$. The commutation relations for the Chevalley basis are given in table 1.

$M = 1$. This is the trivial grading. The entire algebra L is an eigenspace of the elements corresponding to [100], [010], or [001]. They all act as the identity on L.

Table 1. Commutation relations of $sl(3,\mathbb{C})$ generators

	h_1	h_2	e_1	e_2	e_3	e_{-1}	e_{-2}	e_{-3}
h_1	0	0	$2e_1$	$-e_2$	$-e_3$	$-2e_{-1}$	e_{-2}	e_{-3}
h_2	0	0	$-e_1$	$2e_2$	$-e_3$	e_{-1}	$\vert -2e_{-2}$	e_{-3}
e_1	$-2e_1$	e_1	0	e_{-3}	$-e_{-2}$	h_1	0	0
e_2	e_2	$-2e_2$	$-e_{-3}$	0	e_{-1}	0	h_2	0
e_3	e_3	e_3	e_{-2}	$-e_{-1}$	0	0	0	$-h_1-h_2$
e_{-1}	$2e_{-1}$	$-e_{-1}$	$-h_1$	0	0	0	$-e_3$	e_2
e_{-2}	$-e_{-2}$	$2e_{-2}$	0	$-h_2$	0	e_3	0	$-e_1$
e_{-3}	$-e_{-3}$	$-e_{-3}$	0	0	h_1+h_2	$-e_2$	e_1	0

$M=2$. The elements [011], [110] and [101] provide three equivalent Z_2 decompositions. We choose the element [011] and obtain

$$L_0=\{h_1,h_2,e_3,e_{-3}\} \qquad L_1=\{e_1,e_2,e_{-1},e_{-2}\}. \tag{2.5}$$

$M=3$. There are two genuinely inequivalent $\mathbb{Z}_3$ grading of $sl(3,\mathbb{C})$. One is given by the elements [111] and is

$$L_0=\{h_1,h_2\} \qquad L_1=\{e_1,e_2,e_3\} \qquad L_{-1}=\{e_{-1},e_{-2},e_{-3}\}. \tag{2.6}$$

The gradings [011] and [111] are the only two 'coarsest' ones. They are refinements only of the trivial grading. All other gradings are refinements of (2.5), (2.6), or both of the above.

The other $\mathbb{Z}_3$ grading corresponds to the element [012] (or some permutation of $0,1$ and 2), is given as

$$L_0=\{h_1,h_2,e_3,e_{-3}\} \qquad L_1=\{e_1,e_{-2}\} \qquad L_{-1}=\{e_{-1},e_2\}. \tag{2.7}$$

The levels L_1 and L_{-1} in (2.7) are obtained by splitting L_1 of (2.5) into two.

$M=4$. There are two inequivalent gradings of this type. A $\mathbb{Z}_4$ grading corresponds to [211] (and permutations) and is a refinement of the $M=2$ one:

$$L_0=\{h_1,h_2\} \qquad L_1=\{e_1,e_2\} \qquad L_2=\{e_3,e_{-3}\} \qquad L_{-1}=\{e_{-1},e_{-2}\}. \tag{2.8}$$

All permutations of [013] give the same grading as [012].

The second $M=4$ grading is the exceptional one, generated by two elements: $[011]\times[110]$. We have a $\mathbb{Z}_2\times\mathbb{Z}_2$ grading, namely

$$L_{00}=\{h_1,h_2\} \qquad L_{01}=\{e_3,e_{-3}\} \qquad L_{10}=\{e_2,e_{-2}\} \qquad L_{11}=\{e_1,e_{-1}\} \tag{2.9}$$

and it is a refinement of the $\mathbb{Z}_2$ grading.

$M=5$. Two types of elements of order 5 give equivalent $\mathbb{Z}_5$ grading namely [122] and [113] (plus permutations). For [122] we have

$$L_0=\{h_1,h_2\} \qquad L_1=\{e_1,e_2\} \qquad L_2=\{e_{-3}\} \qquad L_{-1}=\{e_{-1}e_{-2}\} \qquad L_{-2}=\{e_3\} \tag{2.10}$$

a refinement of the [211] case. The elements [014] and [023] give $\mathbb{Z}=3$ gradings, equivalent to [012].

$M = 6$. Only one $\mathbb{Z}_6$ grading of this type exists (up to equivalence). We choose it to be [312]:

$$\begin{array}{llll} L_0 = \{h_1, h_2\} & L_1 = \{e_1\} & L_2 = \{e_2\} & L_3 = \{e_3, e_{-3}\} \\ L_{-1} = \{e_{-1}\} & L_{-2} = \{e_{-2}\}. & & \end{array} \tag{2.11}$$

$M = 7$. Up to equivalence we have precisely one such finest grading: the root decomposition of $sl(3, \mathbb{C})$. The element [124] yields:

$$\begin{array}{llll} L_0 = \{h_1, h_2\} & L_1 = \{e_1\} & L_2 = \{e_2\} & L_3 = \{e_{-3}\} \\ L_{-1} = \{e_{-1}\} & L_{-2} = \{e_{-2}\} & L_{-3} = \{e_3\}. & \end{array} \tag{2.12}$$

The entire hierarchy of toroidal gradings of $sl(3, \mathbb{C})$ is presented on figure 1. The arrows indicate mutual refinements. Our particular choice of the elements of finite order, amongst equivalent ones, was such as to make the mutual refinements as explicitly visible as possible. Thus, we choose to keep e_3 and e_{-3} together, whenever possible, rather than choosing e.g. e_1 and e_{-1}.

3. The graded contractions

3.1. General procedure

As outlined in the introduction, we shall introduce a 'contraction matrix' $\varepsilon = \varepsilon^T = \{\varepsilon_{\mu\nu}\} \in \mathbb{C}^{M\times M}$, where M is the order of the grading. The contraction matrix is symmetric (1.5) and its matrix elements must satisfy the Jacobi relations (1.6).

There is a certain arbitrariness in the definition of ε. Whenever we have $[L_\mu, L_\nu] = 0$ in the grading, then $\varepsilon_{\mu\nu}$ is not defined and can be chosen arbitrarily. Furthermore, the Demazure–Tits group [14] acts on the Lie algebra L. It relates different gradings amongst each other and was already used to eliminate 'redundant' gradings corresponding to permutations of s_0, s_1, s_2. Within a given grading their group may permute different grading spaces and thus permute certain rows and columns of the matrix ε. Finally, once the Jacobi conditions (1.6) are solved, it is still possible to normalize some matrix elements in $\varepsilon_{\mu\nu}$ by performing a non-singular change of basis, compatible with the grading. This amounts to the transformation

$$\varepsilon_{\mu\nu} \to \frac{\alpha_\mu \alpha_\nu}{\alpha_\mu + \nu} \varepsilon_{\mu\nu} \tag{3.1}$$

obtained by multiplying each element in the space μ box by a non-zero constant α_μ.

The problem of finding all inequivalent contractions for a given grading of a Lie algebra L thus boils down to several steps:

1. Solve the Jacobi relations (1.6) and obtain all admissible contraction matrices ε. For low values of M, like $\mathbb{Z}_2$ or $\mathbb{Z}_3$ gradings, this is easy to do. For $M \geqslant 4$ it is preferable to solve equations (1.6) in a computer-assisted manner. This was done in the present case for $L \sim sl(3, \mathbb{C})$.

2. Use the Demazure–Tits group to eliminate equivalent matrices ε.

3. Use the change of basis (3.1) to normalize as many non-zero matrix elements $\varepsilon_{\mu\nu}$ as possible.

4. Analyse all limits $\varepsilon_{\mu\nu} \to 0$ for one or more of the matrix elements of the remaining contraction matrices ε, compatible with the Jacobi relations.

The overall procedure can be simplified by starting from the coarsest non-trivial gradings, the $\mathbb{Z} = 2$ one and one of the $\mathbb{Z} = 3$ ones in the present case. All possible inequivalent graded contractions for the coarsest gradings should be found. Then, the gradings should be successively refined and only genuinely new, inequivalent, contractions should be added to those obtained for the coarser grading.

The procedure will now be followed for the $sl(3,\mathbb{C})$ algebra.

At the final level, $M = 7$, the root space grading, we shall recover the results of [6] (and correct some misprints). The contracted algebras were denoted $C_1, \ldots, C_{32}$ and we shall use the same notation to relate the present results to the previous ones.

Note that if we put $k = 0$ in relations (1.6) we obtain

$$\varepsilon_{ij}(\varepsilon_{0i} - \varepsilon_{0j}) = 0 \tag{3.2}$$

Thus, for $\varepsilon_{0i} \neq \varepsilon_{0j}$ we must have $\varepsilon_{ij} = 0$ identically. This means that all contractions for $\varepsilon_{0i} \neq \varepsilon_{0j}$ are discrete.

Without repeating it below, we mention that for each grading we obtain the two trivial contractions $L \to L$ and $L \to nL_1$ (Abelian) respectively by choosing $\varepsilon_{\mu\nu} = 1$, and $\varepsilon_{\mu\nu} = 0$, for all μ, ν.

3.2. *The $M = 2$ grading*

From figure 1 we see that, up to equivalence, only one $M = 2$ grading exists. The contents of the grading spaces 0 and 1 are indicated in figure 1.

We have

$$\varepsilon = \begin{pmatrix} \varepsilon_{00} & \varepsilon_{01} \\ \varepsilon_{01} & \varepsilon_{11} \end{pmatrix} \tag{3.3}$$

and the Jacobi identities are

$$\varepsilon_{01}(\varepsilon_{00} - \varepsilon_{01}) = 0 \qquad \varepsilon_{11}(\varepsilon_{00} - \varepsilon_{01}) = 0. \tag{3.4}$$

After normalization (3.1) we find precisely three inequivalent contractions

$$\varepsilon_1 = \begin{pmatrix} 11 \\ 10 \end{pmatrix} \qquad \varepsilon_2 = \begin{pmatrix} 00 \\ 01 \end{pmatrix} \qquad \varepsilon_3 = \begin{pmatrix} 10 \\ 00 \end{pmatrix}. \tag{3.5}$$

The first leads to the non-decomposable unsolvable algebra C_2 of [6] and table 2 with Levi decomposition

$$C_2 \sim sl(2,\mathbb{C}) \triangleright A_{5,7}(1,-1,-1) \tag{3.6}$$

(the algebra $A_{5,7}(a, b, c)$ is defined in appendix 1, as are other algebras used below).

Matrix ε_2 leads to the non-decomposable nil-potent Lie algebra C_9 of [6]. Matrix ε_3 corresponds to a decomposable unsolvable Lie algebra

$$C_{11} \sim sl(2,\mathbb{C}) \oplus 5A_1. \tag{3.7}$$

The contractions corresponding to ε_1 and ε_2 are continuous ($\varepsilon_{00} = \varepsilon_{01}$, whereas that of ε_3 is discrete ($\varepsilon_{00} \neq \varepsilon_{01}$).

3.3. *The $M = 3$, (012) grading*

We have L_0, L_1, L_{-1} as in figure 1. The contraction matrix is

$$\varepsilon = \begin{pmatrix} \varepsilon_{00} & \varepsilon_{01} & \varepsilon_{0-1} \\ \varepsilon_{01} & \star & \varepsilon_{1-1} \\ \varepsilon_{0-1} & \varepsilon_{1-1} & \star \end{pmatrix} \tag{3.8}$$

Table 2. Lie algebras obtained by continuous (C) and discrete (D) toroidal contractions of $sl(3, \mathbb{C})$. See section 3.10 for explanations

#	Type	Isomorphism class	Grading								
			1	2	3 [111]	3 [012]	4 [211]	4 [011]×[110]	5	6	7
C_1	A	$sl(3, \mathbb{C})$	C								
C_2	Nondecomposable not solvable	$sl(2, \mathbb{C}) \triangleright A_{5,7}^{(1,-1,-1)}$		C							
C_3	B	$NR : 6A_1$			C(0/5) D(1/5)		D(2/5)	D(2/5)	D(3/5)	D(4/5)	D(5/5)
C_4	Nondecomposable	$NR : A_{3,1} \oplus 3A_1$				C(0/3) D(1/3)			D(2/3)		D(3/3)
C_5	solvable	$NR : A_{5,1} \oplus A_1$					C(0/1)		D(1/1)		
C_6	non-nil-potent	$NR : A_{3,1} \oplus A_{3,1}$					C(0/1)	C(0/1)	D(1/1)		
C_7		$NR : A_{6,3}$			C						
C_8	C	$(8, 2, 0)(820)(2, 8)$			C						
C_9	Nondecomposable	$(840)(840)(4, 8)$		C							
C_{10}	nil-potent	$(850)(8520)(258)$			C						
C_{11}	D	$sl(2, \mathbb{C}) \oplus 5A_1$		D							
C_{12}	Decomposable not solvable	$sl(2, \mathbb{C}) \triangleright A_{3,5}(1) \oplus 2A_1$				D					

Table 2. (Continued)

#	Type	Isomorphism class	Grading 1	2	3 [111]	3 [012]	4 [211]	4 [011]×[110]	5	6	7
C_{13}	E	$A(7) \oplus A_1$							D(2/4)	D(3/4)	D(4/4)
C_{14}	Decomposable	$B(7) \oplus A_1$							D(1/2)		D(2/2)
C_{15}	solvable	$C(7) \oplus A_1$							D		
C_{16}		$A(6) \oplus 2A_1$					D(1/3)	D(1/3)	D(2/3)	D(3/3)	
C_{17}		$B(6) \oplus 2A_1$					D(0/1)		D(1/1)		
C_{18}		$A_{5,33}^{(a,b)} \oplus 3A_1$			D(0/2)				D(1/2)	D(2/2)	
C_{19}		$A_{5,38} \oplus 3A_1$							D		
C_{20}		$A_{5,1} \oplus A_{2,1} \oplus A_1$							D		
C_{21}		$A_{5,33}^{(a,b)} \oplus A_{3,1}$							D(1/2)		D(2/2)
C_{22}		$A_{5,38} \oplus A_{3,1}$							D		
C_{23}		$A_{3,1} \oplus A_{2,1} \oplus A_{2,1} \oplus A_1$					D				
C_{24}		$A_{3,1} \oplus A_{2,1} \oplus 3A_1$							D		
C_{25}		$A_{2,1} \oplus A_{2,1} \oplus 4A_1$					D				
C_{26}		$A_{2,1} \oplus 6A_1$							D		
$\tilde{C}_{33}$		$A_{3,5}(-1) \oplus A_{3,5}(-1) \oplus 2A_1$					D				
$\tilde{C}_{34}$		$A_{3,5}(-1) \oplus 5A_1$					D	D			
C_{27}	F	$D(7) \oplus A_1$					C	C			
C_{28}	Decomposable	$A_{6,3} \oplus 2A_1$			C						
C_{29}	nil-potent	$A_{5,1} \oplus 3A_1$					C				
C_{30}		$A_{3,1} \oplus A_{3,1} \oplus 2A_1$					C	C			
C_{31}		$A_{3,1} \oplus 5A_1$					C	C			
C_{32}		$8A_1$	C								

satisfying

$$\begin{aligned} &\varepsilon_{01}(\varepsilon_{00}-\varepsilon_{01})=0 \qquad \varepsilon_{1-1}(\varepsilon_{01}-\varepsilon_{0-1})=0 \\ &\varepsilon_{0-1}(\varepsilon_{00}-\varepsilon_{0-1})=0 \qquad \varepsilon_{1-1}(\varepsilon_{00}-\varepsilon_{01})=0. \end{aligned} \tag{3.9}$$

The asterisks in equation (3.8) signify that the elements of $sl(3,\mathbb{C})$ in the corresponding positions commute, so the contraction parameters (in this case ε_{11} and ε_{-1-1}) are not defined.

From figure 1 we see that the (012) grading is a refinement of the (011) grading, considered above. To obtain new contracted algebras with respect to the $M=2$ case, we must impose

$$\varepsilon_{01} \neq \varepsilon_{0-1}. \tag{3.10}$$

The new contractions will hence all be discrete. From equation (3.8) we have $\varepsilon_{1-1}=0$. This leads to a single new contraction,occurring for

$$\varepsilon_{00}=\varepsilon_{01}\neq 0 \qquad \varepsilon_{0-1}=0 \tag{3.11}$$

(or equivalently $\varepsilon_{00}=\varepsilon_{0-1}\neq 0, \varepsilon_{01}=0$). The contracted algebra is decomposable and unsolvable, namely

$$C_{12} \sim \mathrm{aff}(2,\mathbb{C}) \oplus 2A_1.$$

3.4. The $M=3$, (111) *grading*

The second $M=3$ grading is not equivalent to the first one. Moreover, it is not a refinement of the $M=2$ grading. The contraction matrix is

$$\varepsilon = \begin{pmatrix} \star & \varepsilon_{01} & \varepsilon_{0-1} \\ \varepsilon_{01} & \varepsilon_{11} & \varepsilon_{1-1} \\ \varepsilon_{0-1} & \varepsilon_{1-1} & \varepsilon_{-1-1} \end{pmatrix} \tag{3.12}$$

satisfying

$$\begin{aligned} &\varepsilon_{11}(\varepsilon_{01}-\varepsilon_{0-1})=0 \qquad \varepsilon_{11}\varepsilon_{-1-1}-\varepsilon_{01}\varepsilon_{1-1}=0 \\ &\varepsilon_{-1-1}(\varepsilon_{01}-\varepsilon_{0-1})=0 \qquad \varepsilon_{11}\varepsilon_{-1-1}-\varepsilon_{0-1}\varepsilon_{1-1}=0 \\ &\varepsilon_{1-1}(\varepsilon_{01}-\varepsilon_{0-1})=0. \end{aligned} \tag{3.13}$$

Continuous contractions are obtained for $\varepsilon_{01}=\varepsilon_{0-1}\neq 0$, $\varepsilon_{11}\varepsilon_{-1-1}=\varepsilon_{01}\varepsilon_{1-1}$, discrete ones otherwise. The different possibilities are (see appendix 1 for further explanation):

(1) $\varepsilon_{01}=\varepsilon_{0-1}=1$, $\varepsilon_{11}=1$, $\varepsilon_{-1-1}=\varepsilon_{1-1}=0$. We obtain the non-decomposable, solvable, non-nil-potent Lie algebra C_7 with a six dimensional non-decomposable nil-radical $A_{6,3}$.

(2) $\varepsilon_{01}=\varepsilon_{0-1}=1$, $\varepsilon_{11}=\varepsilon_{1-1}=\varepsilon_{-1-1}=0$. We obtain a special case of the non-decomposable solvable non-nil-potent algebra C_3, with an Abelian nil-radical $6A_1$. The algebra C_3 of [6] depends on five parameters; here they are all set equal to $\varepsilon_{0i}=1, i=\pm1,\pm2,\pm3$.

(3) $\varepsilon_{01}=\varepsilon_{0-1}=0$, $\varepsilon_{11}=\varepsilon_{-1-1}=0$, $\varepsilon_{1-1}=1$. We obtain the non-decomposable nil-potent Lie algebra C_8 with

$$\mathrm{DS}:(8,2,0) \qquad \mathrm{CS}:(8,2,0) \qquad \mathrm{US}:(2,8).$$

Here and below DS, CS and US stand for derived series, central series, and upper central series, respectively [14–16].

(4) $\varepsilon_{01}=\varepsilon_{0-1}=0$, $\varepsilon_{11}=0$, $\varepsilon_{-1-1}=1, \varepsilon_{1-1}=1$. We obtain the non-decomposable nil-potent Lie algebra C_{10} with

$$\mathrm{DS}:(8,5,0) \qquad \mathrm{CS}:(8,5,2,0) \qquad \mathrm{US}:(2,5,8).$$

(5) $\varepsilon_{01} = \varepsilon_{0-1} = 0$, $\varepsilon_{11} = 0$, $\varepsilon_{1-1} = 0, \varepsilon_{-1-1} = 1$. The contracted algebra is decomposable and nil-potent, namely

$$C_{28} = A_{6,3} \oplus 2A_1.$$

The discrete contractions obtained in this case are:

(6) $\varepsilon_{01} = 1$, $\varepsilon_{0-1} \neq 0, 1$, $\varepsilon_{11} = \varepsilon_{-1-1} = \varepsilon_{1-1} = 0$. The contracted algebra is non-decomposable, solvable, non-nil-potent, namely a special case of C_3, depending on one parameter, rather than 5.

(7) $\varepsilon_{01} = 1$, $\varepsilon_{0-1} = 0$, $\varepsilon_{11} = \varepsilon_{-1-1} = \varepsilon_{1-1} = 0$. We obtain a decomposable, solvable, non-nil-potent algebra, a special case of C_{18} (with $a = b = 1$, see [6]).

3.5. *The* $M = 4$, (211) *grading*

For the contents of L_0, L_1, L_2 and L_{-1} see figure 1. The contraction matrix is

$$\varepsilon = \begin{pmatrix} \star & \varepsilon_{01} & \varepsilon_{02} & \varepsilon_{0-1} \\ \varepsilon_{01} & \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{1-1} \\ \varepsilon_{02} & \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{2-1} \\ \varepsilon_{0-1} & \varepsilon_{1-1} & \varepsilon_{2-1} & \varepsilon_{-1-1} \end{pmatrix} \tag{3.14}$$

and the Jacobi identities imply

$$\begin{aligned} &\varepsilon_{11}(\varepsilon_{01} - \varepsilon_{02}) = 0 \qquad \varepsilon_{2-1}(\varepsilon_{01} - \varepsilon_{0-1}) = 0 \\ &\varepsilon_{-1-1}(\varepsilon_{02} - \varepsilon_{0-1}) = 0 \qquad \varepsilon_{2-1}(\varepsilon_{01} - \varepsilon_{02}) = 0 \\ &\varepsilon_{1-1}(\varepsilon_{01} - \varepsilon_{0-1}) = 0 \qquad \varepsilon_{2-1}(\varepsilon_{02} - \varepsilon_{0-1}) = 0 \\ &\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}\varepsilon_{1-1} = 0 \qquad \varepsilon_{11}\varepsilon_{2-1} - \varepsilon_{1-1}\varepsilon_{02} = 0 \\ &\varepsilon_{22}\varepsilon_{-1-1} - \varepsilon_{2-1}\varepsilon_{1-1} = 0 \qquad \varepsilon_{12}\varepsilon_{2-1} - \varepsilon_{22}\varepsilon_{01} = 0 \\ &\varepsilon_{11}\varepsilon_{2-1} - \varepsilon_{1-1}\varepsilon_{01} = 0 \qquad \varepsilon_{12}\varepsilon_{-1-1} - \varepsilon_{1-1}\varepsilon_{0-1} = 0 \\ &\varepsilon_{-1-1}\varepsilon_{12} - \varepsilon_{1-1}\varepsilon_{02} = 0 \qquad \varepsilon_{12}\varepsilon_{2-1} - \varepsilon_{22}\varepsilon_{0-1} = 0. \end{aligned} \tag{3.15}$$

Let us again start from the contractions, satisfying $\varepsilon_{01} = \varepsilon_{02} = \varepsilon_{0-1} = \lambda$, where we can normalize $\lambda = 1$, or $\lambda = 0$

(a) $\lambda = 1$. Relations (3.14) then reduce to

$$\varepsilon_{1-1} = \varepsilon_{11}\varepsilon_{2-1} = \varepsilon_{12}\varepsilon_{-1-1} \qquad \varepsilon_{22} = \varepsilon_{12}\varepsilon_{2-1}. \tag{3.16}$$

For $\varepsilon_{1-1} \neq 0$, $\varepsilon_{22} \neq 0$ no contraction occurs. For $\varepsilon_{1-1} = 0$, $\varepsilon_{22} = 1$ we re-obtain the non-decomposable unsolvable algebra C_2 of table 1 that has occurred already for the $M = 2$ grading. In all other cases we obtain non-decomposable solvable non-nil-potent algebras. Depending on the values of ε_{12}, ε_{2-1}, ε_{11} and ε_{-1-1} we obtain special cases of C_3, C_4, C_5, C_6 and also re-obtain C_7 (see table 2).

(b) $\lambda = 0$. Relations (3.14) in this case reduce to

$$\begin{aligned} &\varepsilon_{11}\varepsilon_{2-1} = 0, \qquad \varepsilon_{12}\varepsilon_{2-1} = 0 \qquad \varepsilon_{12}\varepsilon_{-1-1} = 0 \\ &\varepsilon_{11}\varepsilon_{22} = \varepsilon_{12}\varepsilon_{1-1} \qquad \varepsilon_{22}\varepsilon_{-1-1} = \varepsilon_{2-1}\varepsilon_{1-1}. \end{aligned} \tag{3.17}$$

All contracted algebras are nil-potent and in this way we re-obtain algebras C_8, C_9, C_{10}, and C_{28} and the new algebras C_{27}, C_{29}, C_{30}, and C_{31} of table 2.

All other contractions are discrete. The cases that occur are:

(c)

$$\varepsilon = \begin{pmatrix} \star & a & b & b \\ a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}. \tag{3.18}$$

The algebras obtained are special cases of C_4 and C_{17} and C_{23}.

(d)

$$\varepsilon - \begin{pmatrix} \star & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.19}$$

The contracted algebra is C_{11}, already obtained for $M = 2$.

(e)

$$\varepsilon = \begin{pmatrix} \star & a & b & c \\ a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix}. \tag{3.20}$$

For $abc \neq 0$ we obtain C_3 (with two free parameters out of three). Other algebras obtained are C_{16} (a special case) and C_{25}. We also obtain two algebras overlooked in [6]. Namely, for $a = c = 0$, $b = 1$ we obtain $\tilde{C}_{34}$ of table 2:

$$\tilde{C}_{34} \sim A_{3,5}(-1) + 5A_1 \sim \{h_1 + h_2, e_3, e_{-3}\} \oplus \{h_1 - h_2\} \oplus e_1 \oplus e_{-1} \oplus e_2 \oplus e_{-2}.$$

For $a = c = 1$, $b = 0$ we obtain $\tilde{C}_{33}$ of table 1, namely

$$\begin{aligned} \tilde{C}_{33} \sim A_{3,5}(-1) \oplus A_{3,5}(-1) \oplus 2A_1 \sim \{h_1 + h_2, e_1, e_{-1}\} \\ \oplus\{h_1 + 2h_2, e_2, e_{-2}\} \oplus \{e_3\} \oplus \{e_{-3}\}. \end{aligned}$$

Notice that the $M = 2$ contractions are recovered by coarsening this grading, i.e. putting

$$\varepsilon_{02} = \varepsilon_{22} \qquad \varepsilon_{11} = \varepsilon_{-1-1} = \varepsilon_{1-1} \qquad \varepsilon_{01} = \varepsilon_{0-1} = \varepsilon_{12} = \varepsilon_{2-1}. \tag{3.21}$$

3.6. The $M = 4, [011] \times [110]$ grading ($Z_2 \times Z_2$ grading)

We have a refinement of the $M = 2$ contractions. From the Jacobi relations we find that the contraction matrix ε must have one of the forms

$$\varepsilon_1 = \begin{pmatrix} \star & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \varepsilon_2 = \begin{pmatrix} \star & a & b & c \\ a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix}$$

$$\varepsilon_3 = \begin{pmatrix} \star & 1 & 1 & 1 \\ 1 & vw & v & w \\ 1 & v & vz & z \\ 1 & W & z & wz \end{pmatrix} \qquad \varepsilon_4 = \begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & u & v & w \\ 0 & v & y & z \\ 0 & w & z & t \end{pmatrix}$$

with $vz = wz = vw = 0yw = uz = vt$. The $M = 2$ contractions correspond to

$$\begin{aligned} &\varepsilon_{00,00} = \varepsilon_{00,01} = \varepsilon_{01,01} \qquad \varepsilon_{00,10} = \varepsilon_{00,11} = \varepsilon_{01,01} = \varepsilon_{01,11} \\ &\varepsilon_{10,10} = \varepsilon_{10,11} = \varepsilon_{11,11}. \end{aligned} \tag{3.22}$$

Thus, ε_1 leads to C_{11}, already obtained for the $M = 2$ grading.

The contraction matrix ε_2 leads a special case of C_3 for $abc \neq 0$ (with two free parameters, rather than five as the general case). For $bc \neq 0$, $a = 0$ we have a special case of C_{16} (with one parameter out of three). For $a = b = 0$, $c = 1$ we have algebra $\tilde{C}_{34}$.

The matrix ε_3 for $v = w = 1$, $z = 0$ gives C_2, already obtained for $M = 2$. For $v = 0$, $w = 0$, $z = 1$ we obtain a special case of C_6.

The matrix ε_4 provides the following contractions: $C_8(z = w = v = 0, u = y = t = 1)$, $C_9(z = w = t = 0, u = v = y = 1)$, $C_{27}(z = w = t = u = 0v = y = 1)$, $C_{30}(v = w = z = u = 0, y = t = 1)$ and $C_{31}(v = w = z = u = y = 0, t = 1)$.

3.7. The $M = 5(122)$ grading

The $M = 5$ grading is a refinement of the $M = 4$ one and also of $M = 3(111)$. The spaces L_0, $L_{\pm 1}$, $L_{\pm 2}$ are described in figure 1. The contraction matrix is

$$\varepsilon = \begin{pmatrix} \star & \varepsilon_{01} & \varepsilon_{02} & \varepsilon_{0-2} & \varepsilon_{0-1} \\ \varepsilon_{01} & \varepsilon_{11} & \star & \varepsilon_{1-2} & \varepsilon_{1-1} \\ \varepsilon_{02} & \star & \star & \varepsilon_{2-2} & \varepsilon_{2-1} \\ \varepsilon_{0-2} & \varepsilon_{1-2} & \varepsilon_{2-2} & \star & \star \\ \varepsilon_{0-1} & \varepsilon_{1-1} & \varepsilon_{2-1} & \star & \varepsilon_{-1-1} \end{pmatrix} \tag{3.23}$$

and the Jacobi identities imply

$$\begin{array}{ll} \varepsilon_{1-2}(\varepsilon_{01} - \varepsilon_{0-2}) = 0 & \varepsilon_{2-1}(\varepsilon_{01} - \varepsilon_{02}) = 0 \\ \varepsilon_{1-2}(\varepsilon_{01} - \varepsilon_{0-1}) = 0 & \varepsilon_{2-1}(\varepsilon_{01} - \varepsilon_{0-1}) = 0 \\ \varepsilon_{1-2}(\varepsilon_{0-2} - \varepsilon_{0-1}) = 0 & \varepsilon_{2-1}(\varepsilon_{02} - \varepsilon_{0-1}) = 0 \\ \varepsilon_{1-1}(\varepsilon_{01} - \varepsilon_{02}) = 0 & \varepsilon_{2-2}(\varepsilon_{01} - \varepsilon_{0-1}) = 0 \\ \varepsilon_{1-1}(\varepsilon_{01} - \varepsilon_{0-1}) = 0 & \varepsilon_{2-2}(\varepsilon_{02} - \varepsilon_{0-2}) = 0 \\ \varepsilon_{1-1}(\varepsilon_{0-2} - \varepsilon_{0-1}) = 0 & \varepsilon_{11}(\varepsilon_{01} - \varepsilon_{02}) = 0 \\ \varepsilon_{1-1}(\varepsilon_{02} - \varepsilon_{0-1}) = 0 & \varepsilon_{-1-1}(\varepsilon_{0-2} - \varepsilon_{0-1}) = 0 \\ \varepsilon_{1-1}(\varepsilon_{02} - \varepsilon_{0-2}) = 0 & \\ \varepsilon_{01}\varepsilon_{1-1}\varepsilon_{11}\varepsilon_{2-1} = 0 & \varepsilon_{01}\varepsilon_{2-2} - \varepsilon_{1-2}\varepsilon_{2-1} = 0 \\ \varepsilon_{0-2}\varepsilon_{1-1} - \varepsilon_{1-2}\varepsilon_{-1-1} = 0 & \varepsilon_{02}\varepsilon_{1-1} - \varepsilon_{11}\varepsilon_{2-1} = 0 \\ \varepsilon_{0-1}\varepsilon_{2-2} - \varepsilon_{1-2}\varepsilon_{2-1} = 0 & \varepsilon_{0-1}\varepsilon_{1-1} - \varepsilon_{1-2}\varepsilon_{-1-1} = 0 \\ \varepsilon_{11}\varepsilon_{2-2} - \varepsilon_{1-2}\varepsilon_{1-1} = 0 & \varepsilon_{-1-1}\varepsilon_{2-2} - \varepsilon_{2-1}\varepsilon_{1-1} = 0. \end{array} \tag{3.24}$$

To avoid algebras already obtained for $M = 4$, we must put

$$\varepsilon_{02} \neq \varepsilon_{0-2}. \tag{3.25}$$

Continuous contractions are obtained for $\varepsilon_{01} = \varepsilon_{02} = \varepsilon_{0-2} = \varepsilon_{0-1} = \lambda$, $\lambda = 1$, or $\lambda = 0$, hence they were already obtained for $M = 4$.

Analysing equation (3.23) and keeping in mind (3.24), we see that only the following distinct contraction matrices are allowed

$$\varepsilon_1 = \begin{pmatrix} & \star & a & a & b & a \\ a & 0 & \star & 0 & 0 & \\ a & \star & \star & 0 & 1 & \\ b & \star & 0 & \star & \star & \\ a & 0 & 1 & \star & 0 & \end{pmatrix} \qquad \varepsilon_2 = \begin{pmatrix} \star & a & a & b & c \\ a & 1 & \star & 0 & 0 \\ a & \star & \star & 0 & 0 \\ b & 0 & 0 & \star & \star \\ c & 0 & 0 & \star & 0 \end{pmatrix}$$
$$\varepsilon_3 = \begin{pmatrix} \star & a & a & b & b \\ a & 1 & \star & 0 & 0 \\ a & \star & \star & 0 & 0 \\ b & 0 & 0 & \star & \star \\ b & 0 & 0 & \star & 1 \end{pmatrix} \qquad \varepsilon_4 = \begin{pmatrix} \star & a & b & c & d \\ a & 0 & \star & 0 & 0 \\ b & \star & \star & 0 & 0 \\ c & 0 & 0 & \star & \star \\ d & 0 & 0 & \star & 0 \end{pmatrix}. \tag{3.26}$$

In ε_1, ε_2 and ε_3 we have $a \neq b$, in ε_4 we have $b \neq c$.

The case ε_1 leads to algebras C_5, C_{15} and C_{20} (for $a \neq 0$, $b = 0$; $a = 1$, $b = 0$; and $a = 0$, $b = 1$, respectively).

The matrix ε_2 leads to the algebras C_4 (with two out of three parameters, $abc \neq 0$), C_{14} ($ac \neq 0$, $b = 0$, one parameter out of two), C_{17} ($b = 0$, $ac \neq 0$), C_{21} ($a = 0$, $bc \neq 0$, one parameter out of two), C_{19}($a = 1$, $b = c = 0$) and C_{24} ($a = c = 0$, $b = 1$).

The matrix ε_3 leads to the general case of the algebra C_6 (one parameter, $ab \neq 0$). For $a = 1$, $b = 0$ (or $a = 0$, $b = 1$) we obtain the algebra $C_{22} \sim A_{5,38} \oplus A_{3,1}$ (due to a misprint in [6], C_{22} was identified as $A_{5,38} \oplus 3A_1$).

For ε_4 we obtain special cases of C_3, C_{13}, C_{16}, and C_{18} and also the general case of C_{26} and $\tilde{C}_{34}$

3.8. The $M = 6[312]$ grading

The $M = 6$ grading is a refinement of both $M = 4$ gradings, as well as one of the $M = 3$ gradings. Seven different contraction matrices $\varepsilon_1 \ldots \varepsilon_7$ are allowed by the Jacobi identities. We shall not spell them out here. The contents of the individual grading levels are shown in figure 1.

No new classes of contracted algebras are obtained at this level. Class 4 C_3 makes its reappearance, this time with four free parameters out of five possible ones. Similarly, for C_{13} we obtain one more parameter (three out of four), for C_{16} and C_{19} we obtain the general cases with three and two parameters, respectively.

3.9. The $M = 7[124]$ grading

This grading, the finest toroidal one, was studied in detail earlier [6]. On its own, it yields all toroidal contractions, corresponding to seven different types of contraction matrices.

However, the $M = 7$ grading is a refinement of the $M = 6$, and $M = 5$ ones and as such, adds very little to the picture we already have.

Indeed, for $M = 7$ we re-obtain the algebras C_3, C_4, C_{13}, C_{14} and C_{21}, this time with the complete set of five, three, four, two and two parameters, respectively.

3.10. Summary of results

Table 2 contains a list of all Lie algebras obtained by one of the graded contractions of $sl(3, \mathbb{C})$. In column 1 we repeat the symbol $C_1, \ldots, C_{32}$ given to the algebras in [6] (algebras $\tilde{C}_{33}$ and $\tilde{C}_{34}$ were omitted). In columns 2 and 3 we characterize the isomorphism class of the Lie algebra obtained, following principles outlined elsewhere [16]. Algebras of type A, B and C are non-decomposable, D, E and F are explicitly decomposed into direct sums. Non-trivial Levi decompositions [16–18] are denoted $S \triangleright R$, where S is semisimple (actually simple and equal to $sl(2, \mathbb{C})$ in our case) and R is the radical (maximal solvable ideal). For type B algebras (non-decomposable, solvable, but not nil-potent), we identify the nil-radical (maximal nil-potent ideal) in column 3. For type C algebras (non-decomposable nil-potent) we give the derived series, lower central series and upper central series [16–18] in Column 3.

All Lie algebras of dimension $d \leqslant 5$ have been classified; nil-potent Lie algebras have been classified up to dimension $d = 6$ (see [19] and references therein). The notations $A_{p,q}$ (e.g. $A_{3,1}$ or $A_{5,7}(1, -1, -1)$) were introduced earlier [19]. The first label denotes the dimension of the Lie algebra, the second simply enumerates different isomorphism classes of Lie algebras of the same dimension. Some of the representative Lie algebras depend on continuous parameters. These are put in brackets, e.g. $A_{5,7}(a, b, c)$. The algebras $A(7)$, $B(7)$, $C(7)$, $A(6)$, $B(6)$ and $D(6)$ are defined in appendix 1.

In the last column we specify the lowest grading at which a given Lie algebra first occurs. The letter C indicates a continuous contraction, D a discrete one.

The Lie algebras obtained by the contractions can depend on continuous parameters, up to five of them, as a matter of fact. The parameters are non-zero values of the contraction coefficients $\varepsilon_{\mu\nu}$ that figure in a non-removable way in the commutation relations. Thus, they cannot be annulled, or normalized to some chosen number by a change of basis after the contraction. It turns out that all parameters have the same origin. They are values of the contraction coefficients $\varepsilon_{0\mu}(\mu = \pm 1, \pm 2, \pm 3)$, i.e. they are directly related to eigenvalues of the operators h_1 and h_2. For continuous contractions we have $\varepsilon_{0\mu} = \lambda$, with $\lambda = 0$, or λ normalizable to $\lambda = 1$. Hence, continuous parameters occur in discrete contractions only.

From table 2 we see that all continuous contractions correspond to gradings with $M \leqslant 4$. What occurs for $M = 5$ is that some new discrete contractions occur, always leading to solvable decomposable Lie algebras. Typically, solvable Lie algebras, depending on parameters, first occur with an incomplete set of parameters. Further refinements of the grading then remove constraints on the parameters.

Nilpotent Lie algebras are all obtained by continuous contractions. The reason for this is that they all correspond to $\varepsilon_{0\mu} = 0$ (for all values of μ). All unsolvable Lie algebras in the list contain an $sl(2, \mathbb{C})$ subalgebra and are obtained for $M \leqslant 3$.

4. Casimir operators of the contracted Lie algebras

4.1. General comments

The Lie algebra $sl(3, \mathbb{C})$ is of rank 2 and as such has two independent Casimir operators, i.e. operators in the enveloping algebra of $sl(3, \mathbb{C})$, spanning the centre of the enveloping algebra (and hence commuting with all elements of the Lie algebra).

The following question arises. What happens to these Casimir operators when the algebra $sl(3, \mathbb{C})$ undergoes a contraction? For continuous contractions we can treat the graded contraction as a singular change of basis. The contraction parameters will then figure in the Casimir operators themselves and we can view their limits by inspection. We must, however, keep in mind that while $sl(3, \mathbb{C})$ has precisely two such operators, the contracted algebras may have more. Indeed, in the extreme case of an Abelian Lie algebra, every basis element of the Lie algebra is a Casimir operator.

For discrete contractions the situation is quite different. No continuous limiting procedure is possible, essentially by definition. Moreover, the concept of a Casimir operator must be generalized, to go beyond polynomials in the generators.

A fruitful way of doing this is to view the Casimir operators as being associated with invariants of the co-adjoint representation of the corresponding Lie group G [19–24]. Such invariants can be calculated directly as follows. Choose a basis for the corresponding Lie algebra L in which the commutation relations are

$$[X_i, X_k] = C_{ikl} X_l \qquad 1 \leqslant i, k, l \leqslant N. \tag{4.1}$$

Represent the operators X_i in the co-adjoint representation by the vector fields

$$\hat{X}_i = -C_{ikl} x_k \frac{\partial}{\partial x_l} \tag{4.2}$$

acting on functions $F(x_1, \ldots, x_N)$. The invariants are obtained as a set of functionally independent solutions of the linear first-order differential equations

$$X_i F(x_1, \ldots, x_N) = 0 \qquad i = 1, \ldots, N. \tag{4.3}$$

If the solutions are polynomials, we obtain the Casimir operators by replacing the variables x_i by the generators X_i and symmetrizing, whenever necessary. If the solutions are rational functions, or more general functions, e.g. transcendental functions, we shall call the corresponding operators 'generalized Casimir operators'.

We have calculated the invariants of the co-adjoint representation directly for all Lie algebras obtained by the contractions. In all cases at least two independent invariants exist. For continuous contractions the invariants are always polynomials in the generators. For discrete ones we sometimes obtain expressions involving arbitrary powers of the generators, not necessarily positive integer ones. Examples will be given below.

The $sl(3,\mathbb{C})$ invariants are of course well known and in the basis we are using we write them as

$$\begin{aligned} C^{(2)} &= h_1^2 + h_2^2 + h_1 h_2 + 3(e_1 e_{-1} + e_2 e_{-2} + e_3 e_{-3}) \\ C^{(3)} &= 3h_1 h_2 (h_1 - h_2) + 2(h_1^3 - h_2^3) + 9(e_1 e_{-1} - 2e_2 e_{-2} + e_3 e_{-3}) h_1 \\ &\quad + 9(2e_1 e_{-1} - e_2 e_{-2} - e_3 e_{-3}) h_2 + 27(e_1 e_2 e_3 + e_{-1} e_{-2} e_{-3}). \end{aligned} \tag{4.4}$$

Note that these are invariants of the co-adjoint representation so that h_i and e_μ are commuting variables. We shall not perform the symmetrization needed to obtain the actual Casimir operators.

4.2. Casimir operators for continuous contractions

Let us run through all continuous contractions in table 2, following the degree of the grading.

$M = 2$. We go to a new basis by multiplying all elements of the level 0 $\{h_1, h_2, e_3, e_{-3}\}$ by a constant α, those of level 1 $\{e_1, e_{-1}, e_2, e_{-2}\}$ by a constant β. The contraction matrix is then expressed as

$$\varepsilon = \begin{pmatrix} \varepsilon_{00} & \varepsilon_{01} \\ \varepsilon_{01} & \varepsilon_{11} \end{pmatrix} = \begin{pmatrix} \alpha & \alpha \\ \alpha & \frac{\beta^2}{\alpha} \end{pmatrix}. \tag{4.5}$$

The invariants (4.4) in the new basis are

$$\begin{aligned} {}^{(2)} &= \frac{1}{\alpha^2}(h_1^2 + h_2^2 + h_1 h_2 + 3e_3 e_{-3}) + \frac{3}{\beta^2}(e_1 e_{-1} + e_2 e_{-2}) \\ C^{(3)} &= \frac{1}{\alpha^3}\{3h_1 h_2 (h_1 - h_2) + 2(h_1^3 - h_2^3) + 9e_3 e_{-3}(h_1 - h_2)\} \\ &\quad + \frac{1}{\alpha\beta^2}\{9(e_1 e_{-1} - 2e_2 e_{-2}) h_1 + 9(2e_1 e_{-1} - e_2 e_{-2}) h_2 \\ &\quad + 27(e_1 e_2 e_3 + e_{-1} e_{-2} e_{-3})\}. \end{aligned} \tag{4.6}$$

A direct calculation shows that the algebra C_2 of table 2 has two Casimir invariants, correctly obtained in the limit as

$$\begin{aligned} \frac{\beta^2 C^{(2)}}{3} &\underset{\substack{\beta\to 0 \\ \alpha=1}}{\to} I_1 = e_1 e_{-1} + e_2 e_{-2} \\ \frac{\beta^2 C^{(3)}}{9} &\underset{\substack{\beta\to 0 \\ \alpha=1}}{\to} I_2 = e_1 e_{-1}(h_1 + 2h_2) - e_2 e_{-2}(2h_1 + h_2) + 3(e_1 e_2 e_3 + e_{-1} e_{-2} e_{-3}). \end{aligned} \tag{4.7}$$

The other algebra obtained by a continuous contraction in the $M = 2$ case is the non-decomposable nil-potent Lie algebra C_9. It is obtained by putting $\beta^2 = \alpha \to 0$. The algebra C_9 has a four-dimensional centre

$$C(L) = \{h_1, h_2, e_3, e_{-3}\}. \tag{4.8}$$

Thus, all four elements of $C(L)$, i.e. of the level 0 in the grading, are invariants. Taking the limit $\beta^2 = \alpha \to 0$ in $\alpha^2 C^{(2)}$ and $\alpha^3 C^{(3)}$ we obtain two invariants that are polynomials in the generators (4.8). In other words, we obtain only two invariants instead of the four existing ones.

$M = 3$. Only one of the two $M = 3$ gradings provides continuous graded contractions, namely the [111] grading. To obtain these contractions as singular changes of basis, we multiply the elements of $L_0 = \{h_1, h_2\}$, $L_1 = \{e_1, e_2, e_3\}$ and $L_{-1} = \{e_{-1}, e_{-2}, e_{-3}\}$, by the constants α, β, and γ, respectively. The contraction matrix is

$$\begin{pmatrix} \star & \varepsilon_{01} & \varepsilon_{0-1} \\ \varepsilon_{01} & \varepsilon_{11} & \varepsilon_{1-1} \\ \varepsilon_{0-1} & \varepsilon_{1-1} & \varepsilon_{-1-1} \end{pmatrix} = \begin{pmatrix} \alpha & \alpha & \alpha \\ \alpha & \frac{\beta^2}{\gamma} & \frac{\beta\gamma}{\alpha} \\ \alpha & \frac{\beta\gamma}{\alpha} & \frac{\gamma^2}{\beta} \end{pmatrix}. \tag{4.9}$$

The $sl(3, \mathbb{C})$ invariants in the new basis are

$$\begin{aligned} C^{(2)} &= \frac{1}{\alpha^2}(h_1^2 + h_2^2 + h_1 h_2) + \frac{3}{\beta\gamma}(e_1 e_{-1} + e_2 e_{-2} + e_3 e_{-3}) \\ C^{(3)} &= \frac{1}{\alpha^3}\{3h_1 h_2(h_1 - h_2) + 2(h_1^3 - h_2^3)\} \\ &\quad + \frac{9}{\alpha\beta\gamma}\{(e_1 e_{-1} - 2e_2 e_{-2} + e_3 e_{-3})h_1 + (2e_1 e_{-1} - e_2 e_{-2} - e_3 e_{-3})h_2\} \\ &\quad + \frac{27}{\beta^3} e_1 e_2 e_3 + \frac{27}{\gamma^3} e_{-1} e_{-2} e_{-3}. \end{aligned} \tag{4.10}$$

Let us now consider the algebras of table 2, obtained by continuous contractions for this grading.

Algebra C_3 (with all parameters $\varepsilon_{0i} = 1$, $i = \pm 1, \pm 2, \pm 3$). We must set

$$\alpha = 1, \beta = \gamma^p \qquad \tfrac{1}{2} < p < 2, \gamma \to 0 \tag{4.11}$$

to obtain $\varepsilon_{11} = \varepsilon_{1-1} = \varepsilon_{-1-1} \to 0$.

A direct calculation shows that C_3 has four functionally independent invariants. A suitable basis is

$$I_1 = e_1 e_{-1} \qquad I_2 = e_2 e_{-2} \qquad I_3 = e_3 e_{-3} \qquad I_4 = e_1 e_2 e_3. \tag{4.12}$$

The limiting procedure for the invariants $C^{(2)}$, $C^{(3)}$ of equation (4.10) provides the invariants $I_1 + I_2 + I_3$, I_4 (for $1 < p < 2$), and $I_1 I_2 I_3 / I_4$ (for $\frac{1}{2} < p < 1$), i.e. only 3 out of the 4 invariants.

Algebra C_7. The contraction limits are

$$\alpha = 1 \qquad \gamma = \beta^2 \to 0. \tag{4.13}$$

The solvable non-decomposable Lie algebra C_7 has two invariants, correctly obtained in the limit as

$$\begin{aligned} \frac{\beta^3}{3} C^{(2)} &\to I_1 = e_1 e_{-1} + e_2 e_{-2} + e_3 e_{-3} \\ \frac{\beta^6}{27} C^{(3)} &\to I_2 = e_{-1} e_{-2} e_{-3}. \end{aligned} \tag{4.14}$$

Algebra C_8. We must set

$$\alpha = \gamma^{p+1}\beta = \gamma^{p} \qquad \tfrac{1}{2} < p < 1 \qquad \gamma \to 0. \tag{4.15}$$

The non-decomposable nil-potent Lie algebra obtained, C_8, has two invariants, h_1 and h_2. The $sl(3,\mathbb{C})$ Casimir operators correctly contract to two different homogenous polynomials in h_1 and h_2.

Algebra C_{10}. We set

$$\alpha = \gamma^3 \qquad \beta = \gamma^2 \qquad \gamma \to 0. \tag{4.16}$$

The situation is the same as for C_8. Two invariants exist for the contracted algebra: h_1 and h_2; elements of the centre of C_8. The $sl(3,\mathbb{C})$ invariants correctly contract to two different polynomials in h_1 and h_2.

Algebra C_{28}. To obtain this algebra, we set

$$\gamma = \alpha^p \qquad \beta = \gamma^{2p} \qquad \tfrac{1}{3} < p \qquad \alpha \to 0. \tag{4.17}$$

The contracted algebra is nil-potent and decomposable. It has 6 invariants:

$$h_1, h_2, e_1, e_2, e_3 \qquad I = e_1e_{-1} + e_2e_{-2} + e_3e_{-3}. \tag{4.18}$$

From $C^{(2)}$ we obtain two invariants in the $\alpha \to 0$ limit, namely $h_1^2 + h_2^2 + h_1h_2$ for $\frac{1}{3} < p < \frac{2}{3}$, and $e_1e_{-1} + e_2e_{-2} + e_3e_{-3}$, if we choose $\frac{2}{3} < p$.

From $C^{(3)}$ we obtain two more, namely a further polynomial in h_1, h_2 for $\frac{1}{3} < p < \frac{1}{2}$, $e_1e_2e_3$ for $p > \frac{1}{2}$. Thus, 4 out of 6 invariants of C_{28} can be obtained as limits of the two $sl(3,\mathbb{C})$ invariants.

$M = 4$. We need only consider one of the two $M = 4$ gradings, namely the [211] grading, since the other one gives equivalent results. The new basis is obtained by multiplying elements of the zero level $\{h_1, h_2\}$ by α, level 1 $\{e_1, e_2\}$ by β, level 2 $\{e_3, e_{-3}\}$ by γ and level -1 $\{e_{-1}, e_{-2}\}$ by δ. The contraction matrix is

$$\varepsilon = \begin{pmatrix} \star & \varepsilon_{01} & \varepsilon_{02} & \varepsilon_{0-1} \\ \varepsilon_{01} & \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{1-1} \\ \varepsilon_{02} & \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{2-1} \\ \varepsilon_{0-1} & \varepsilon_{1-1} & \varepsilon_{2-1} & \varepsilon_{-1-1} \end{pmatrix} = \begin{pmatrix} \alpha & \alpha & \alpha & \alpha \\ \alpha & \frac{\beta^2}{\gamma} & \frac{\beta\gamma}{\delta} & \frac{\beta\delta}{\alpha} \\ \alpha & \frac{\beta\gamma}{\delta} & \frac{\gamma^2}{\alpha} & \frac{\gamma\delta}{\beta} \\ \alpha & \frac{\beta\delta}{\alpha} & \frac{\gamma\delta}{\beta} & \frac{\delta^2}{\gamma} \end{pmatrix}. \tag{4.19}$$

The $sl(3,\mathbb{C})$ Casimir operators in the new basis are

$$\begin{aligned} C^{(2)} &= \frac{1}{\alpha^2}(h_1^2 + h_2^2 + h_1h_2) + \frac{3}{\beta\delta}(e_1e_{-1} + e_2e_{-2}) + 3\frac{e_3e_{-3}}{\gamma^2} \\ C^{(3)} &= \frac{1}{\alpha^3}\{3h_1h_2(h_1 - h_2) + 2(h_1^3 - h_2^3)\} \\ &\quad + \frac{9}{\alpha\beta\delta}\{(e_1e_{-1} - 2e_2e_{-2})h_1 + (2e_1e_{-1} - e_2e_{-2})h_2\} \\ &\quad + \frac{9}{\alpha\gamma^2}e_3e_{-3}(h_1 - h_2) + \frac{27}{\beta^2\gamma}e_1e_2e_3 + \frac{27}{\gamma\delta^2}e_{-1}e_{-2}e_{-3}. \end{aligned} \tag{4.20}$$

Let us run through the continuous contractions in this case.

Algebra C_4 (with all $\varepsilon_{0i} = 1$). In this case we put

$$\alpha = 1 \qquad \gamma = \delta^2 \qquad \beta = \delta^p \qquad 1 < p < 3 \qquad \delta \to 0. \tag{4.21}$$

The algebra has two invariants both correctly obtained from the $sl(3, \mathbb{C})$ invariants in the $\delta \to 0$ limit, namely

$$I_1 = e_3 e_{-3} \qquad I_2 = e_1 e_2 e_3. \tag{4.22}$$

Algebra C_5 (with all $\varepsilon_{0i} = 1$). We put

$$\alpha = 1\gamma = \delta^p \qquad \beta = \delta^{p+1} \qquad 0 < p < 2 \qquad \delta \to 0. \tag{4.23}$$

Again, two invariants exist, both of which can be obtained as limits of $C^{(2)}$ and $C^{(3)}$, namely

$$I_1 = e_1 e_{-1} + e_2 e_{-2} \qquad I_3 = e_1 e_2 e_3. \tag{4.24}$$

Algebra C_6 (with all $C_{0i} = 1$). We put

$$\alpha = 1 \qquad \beta = \delta \qquad \gamma = \delta^2 \qquad \delta \to 0. \tag{4.25}$$

The algebra C_6 has two invariants, correctly obtained in the $\delta \to 0$ limit as

$$I_1 = e_3 e_{-3} \qquad I_2 = e_3 e_{-3}(h_1 - h_2) + 3(e_1 e_2 e_3 + e_{-1} e_{-2} e_{-3}). \tag{4.26}$$

Algebra C_{27}. To obtain this nil-potent decomposable Lie algebra we put

$$\alpha = \delta^{2p} \qquad \beta = \delta^{p+1} \qquad \gamma = \delta^p \qquad 0 < p < 2. \tag{4.27}$$

The algebras has four invariants, all corresponding to elements of the centre

$$\{h_1, h_2, e_1, e_2\}. \tag{4.28}$$

The limits of the $sl(3, \mathbb{C})$ invariants provide just two of them, namely two polynomials in h_1, and h_2.

Algebra C_{29}. We put

$$\alpha = \delta^q \qquad \beta = \delta^{p+1} \qquad \gamma = \delta^p \qquad 0 < p < 2 \qquad 2p - q > 0 \qquad p - q + 2 > 0. \tag{4.29}$$

The algebra C_{29} has six invariants, five of them in the centre of C_{29}:

$$\{h_1, h_2, e_1, e_2, e_3, I = e_1 e_{-1} + e_2 e_{-2}\}. \tag{4.30}$$

In the $\delta \to 0$ limit $C^{(2)}$ provides two of these invariants: $h_1^2 + h_2^2 + 2h_1 h_2$ for $p + 2 < 2q$ and $e_1 e_{-1} + e_2 e_{-2}$ for $p + 2 > 2q$. The $sl(3, \mathbb{C})$ invariant $C^{(3)}$ also provides two: another polynomial in h_1 and h_2 for $3p + 2 < 3q$ and $e_1 e_2 e_3$ for $3p + 2 > 3q$. We do not, however, obtain e_1, e_2, and e_3 separately.

Algebra C_{30}. We put

$$\begin{gathered} \alpha = \delta^{q+1} \qquad \beta = \delta^q \qquad \gamma = \delta^p \qquad 0 < p < 2 \qquad \delta \to 0 \\ 2q - p > 0 \qquad 2p - q - 1 > 0 \qquad p + q - 1 > 0 \qquad p - q + 1 > 0. \end{gathered} \tag{4.31}$$

The algebra C_{30} has four invariants:

$$\{h_1, h_2, e_3, e_{-3}\} \tag{4.32}$$

all elements of the centre. Three of them, h_1, h_2 and $e_3 e_{-3}$ can be viewed as limits of $C^{(2)}$ and $C^{(3)}$ (for different values of p and q).

Algebra C_{31}. We put

$$\begin{array}{lllll} \alpha = \delta^{2p} & \beta = \delta^{q} & \gamma = \delta^{p} & 0 < p < 2 & \delta \to 0 \\ 2q - p > 0 & p + q - 1 > 0 & -2p + q + 1 > 0 & p - q + 1 > 0. & \end{array} \tag{4.33}$$

The algebra C_{31} has a centre of dimension 6 namely

$$\{h_1, h_2, e_1, e_2, e_{-1}, e_{-2}\}. \tag{4.34}$$

The invariant $C^{(2)}$ yields two invariants namely $h_1^2 + h_2^2 + h_1h_2$ for $4p > q + 1$ and $e_1e_{-1} + e_2e_{-2}$ for $4p < q + 1$. The invariant $C^{(3)}$ yields a complementary expression in h_1 and h_2 for $4p > q + 1$ and a further invariant for $4p < q + 1$.

4.3. Casimir operators for discrete contractions

For discrete contractions no continuous limiting procedure for the Casimir operators exists. The only possibility is to calculate the invariants of the co-adjoint representation directly. That is easy to do and we shall give several examples.

Algebra C_3. Let us consider the $M = 7$ grading. The algebra is solvable with an Abelian nil-radical of dimension $\dim NR(L) = 6$. The case $M = 7$ corresponds to $\varepsilon_{0\mu}$ all different. Solvable Lie algebras of arbitrary dimension with Abelian nil-radicals and their invariants were studied elsewhere [21]. In the case under consideration the algebra has four independent invariants. They can be chosen to be:

$$I_1 = e_1^{\varepsilon_{0-1}} e_{-1}^{\varepsilon_{01}} \qquad I_2 = e_2^{\varepsilon_{0-2}} e_{-2}^{\varepsilon_{02}} \qquad I_3 = e_3^{\varepsilon_{0-3}} e_{-3}^{\varepsilon_{03}} \qquad I_4 = e_1^{\varepsilon_{02}\varepsilon_{03}} e_2^{\varepsilon_{01}\varepsilon_{03}} e_3^{\varepsilon_{01}\varepsilon_{02}} \tag{4.35}$$

where one of the constants, e.g. e_{01} can be normalized to be $\varepsilon_{01} = 1$.

If all $\varepsilon_{0\mu}$ are equal, we can set $\varepsilon_{0\mu} = 1$ and we re-obtain the continuous case with invariants (4.12). Notice that the invariants are rational only if the ratios $\varepsilon_{0\mu}/\varepsilon_{0-\mu}$ and $\varepsilon_{01}/\varepsilon_{02}$, $\varepsilon_{01}/\varepsilon_{03}$ are themselves rational.

Algebra C_{11}. The algebra, obtained by the $M = 2$ discrete contraction, decomposes into $sl(2, \mathbb{C})$ and 5 one-dimensional algebras. The invariants hence are

$$I = (h_1 + h_2)^2 + 4e_3e_{-3}, e_1, e_{-1}, e_2, e_{-2}, h_1 - h_2. \tag{4.36}$$

Algebra C_{19}. This Lie algebra occurs at the level $M = 5$. It is the direct sum of a five-dimensional solvable Lie algebra and 3 one-dimensional ones. The solvable Lie algebra $A_{5,38}$ has a three- dimensional nil-radical that is a Heisenberg algebra. Such algebras were classified for all dimensions and their invariants are known [24]. Thus, the algebra C_{19} has four functionally independent invariants

$$I_1 = e_{-1} \qquad I_2 = e_{-2} \qquad I_3 = e_3 \qquad I_4 = \frac{(h_1 - h_2)e_{-3} + 3e_1e_2}{e_3}. \tag{4.37}$$

Notice that I_4 is rational, but not polynomial, in keeping with the general results for this type of Lie algebra [24].

It is quite easy to calculate the invariants for all other Lie algebras in table 2, but we shall not present the results here.

5. Conclusions

A detailed study of toroidal graded contractions for the Lie algebra $sl(3, \mathbb{C})$ has brought out some general features, which we intend to pursue further. One is the great richness of the results, i.e. the diversity of Lie algebras obtained by different contractions, already for relatively coarse gradings. The finer gradings ($M \geqslant 5$ for $sl(3, \mathbb{C})$) contribute only to discrete contractions and only 'refine' the possible values of parameters that characterize classes of solvable Lie algebras.

The study of all graded contractions of $sl(3, \mathbb{C})$ ordered according to the hierarchy of gradings makes evident three important advantages of the grading preserving approach to the study of contractions:

(1) Besides the 'continuous' solutions of equation (1.6) corresponding to the generalized Wigner–Inonu contractions one also obtains all the 'discrete' solutions of (1.6), i.e. the discrete contractions.

(2) Even if one is interested exclusively in the continuous contractions, there is an important simplification here. Rather than analysing all the singular basis transformations at once, which evidently is a difficult and unruly problem, the graded approach allows one to split that problem into well-defined smaller ones corresponding to each non-equivalent grading. Moreover, a physics problem usually imposes additional restrictions on possible contractions, namely that certain subalgebras should not be deformed during a contraction. The latter requirement is in this approach inplemented in an elementary manner, further restricting the range of gradings one needs to study [9].

(3) Finite gradings of representations of $sl(3, \mathbb{C})$, or of any semisimple Lie algebra, are well known and straightforward to describe [13]. Therefore, the study of graded contractions of representations of a Lie algebra is a well-defined problem [7] similar to the one solved here for (the adjoint representation of) $sl(3, \mathbb{C})$.

Some conclusions can also be drawn from section 4, concerning contractions of Casimir operators. We have seen that at least for continuous contractions, the $sl(3, \mathbb{C})$ invariants always contract to invariants of the contracted Lie algebras. In some cases all the invariants of the contracted algebras are obtained, in others only a subset of them. This is related to a more general and yet unresolved problem, namely the contraction of the universal enveloping algebra of a Lie algebra and its relation to the enveloping algebra of the contracted Lie algebra. Many algebras obtained by the contractions, specially the nil-potent Lie algebras and some of the decomposable ones, acquire centres produced by the contraction. This is the main source of 'missing' invariants. The number of these missing invariants can serve as a measure of the distortion of the original Lie algebra by the contraction. The original applications [1–3] of Lie algebra contractions were to relate different physical theories amongst each other and thus to provide a mathematical tool for the correspondance principle.

Other applications of contractions arise once we are able to implement them analytically, i.e. introduce contractions parameters into a realization of the considered algebras. As an example, consider the problem of separation of variables for a Laplace–Beltrami operator (or Hamilton–Jacobi operator) on a homogeneous space. For simplicity consider a sphere $S_2 \sim O(3)/O(2)$. Two separable coordinate systems exist: spherical and elliptic. In the limit when the radius R of the spheres satisfies $R \to \infty$ we obtain a Euclidean plane $E_2 \sim E(2)/O(2)$ where $E(2)$ is the Euclidean group. Four separable coordinate systems exist on E_2: cartesian, polar, parabolic and elliptic. All of them are recovered in appropriately chosen contraction limits [25]. Indeed, using geodesical coordinates [26], x_1

and x_2 we realize the $o(3)$ algebra as

$$\begin{aligned} \pi_1 &= \frac{\partial}{\partial x_1} + \frac{1}{R^2} x_1 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \\ \pi_2 &= \frac{\partial}{\partial x_2} + \frac{1}{R^2} x_2 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \\ L_3 &= x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \end{aligned} \tag{5.1}$$

with commutation relations

$$[L_3, \pi_1] = \pi_2 \qquad [L_3, \pi_2] = -\pi_1 \qquad [\pi_1, \pi_2] = \frac{L_3}{R^2}. \tag{5.2}$$

For $R < \infty$ we have $o(3)$, for $R \to \infty$ we obtain $e(2)$. Pursuing the contraction analytically, one obtains relations between partial differential equations and the separated ordinary differential equations. Furthermore, one obtains asymptotic formulas for the special functions occurring as solutions [25].

A related application is to symmetries of linear, and specially nonlinear, differential equations [27, 28]. As the simplest example, consider the algebra $sl(2, \mathbb{R})$. Two continuous graded toroidal contractions exist:

$$sl(2, \mathbb{R}) \to p(1, 1) \qquad sl(2, \mathbb{R}) \to A_{3,1} \tag{5.3}$$

where $p(1, 1)$ is the Lie algebra of the Poincaré group in $1 + 1$ dimensions and $A_{3,1}$ is the Heisenberg algebra. The first can be realized by putting

$$\hat{h} = x\partial_x - y\partial_y \qquad \hat{e}_\varepsilon = -x\partial_u + 2\varepsilon u\partial_y \qquad \hat{f}_\varepsilon = y\partial_u - e\varepsilon u\partial_x. \tag{5.4}$$

The commutation relations are

$$[\hat{h}, \hat{e}_\varepsilon] = \hat{e}_\varepsilon, [\hat{h}, \hat{f}_\varepsilon] = -f_\varepsilon, [e_\varepsilon, f_\varepsilon] = 2\varepsilon h \tag{5.5}$$

i.e. $sl(2, \mathbb{R})$ for $\varepsilon \neq 0$, $p(1, 1)$ for $\varepsilon \to 0$. This can be used to study relations between $SL(2, \mathbb{R})$ and $P(1, 1)$ invariant partial differential equations, if we consider u to be a dependent variable, x and y independent ones.

Choosing

$$h_\alpha = \tfrac{1}{2}(\alpha x + 1)\partial_x - \tfrac{1}{2}\alpha y\partial_y \qquad e_\alpha = (\alpha x + 1)\partial_y \qquad f = y\partial_x \tag{5.6}$$

we have $sl(2, \mathbb{R})$ commutation relations far $\alpha \neq 0$, $A_{3,1}$ ones for $\alpha \to 0$. Again this can be used to relate $SL(2, \mathbb{R})$ invariant ordinary differential equations to equations invariant under the Heisenberg group. In this case y is to be considered as a dependent variable, x an independent one.

Finally, let us add a few words on possible applications of the contractions of the Lie algebra $sl(3, \mathbb{C})$ studied in this article.

The first is somewhat speculative and concerns condensed matter physics. The real form $SL(3, \mathbb{R})$ of the group $SL(3, \mathbb{C})$ occurs as the symmetry group of the constitutive functionals of certain simple materials [29]. If the conditions under which this symmetry pertains are relaxed, or modified, e.g. by placing the materials in external fields, the symmetry group will be changed. In particular, it may be reduced to a subgroup of $SL(3, \mathbb{R})$, as investigated by Nono [29]. On the other hand, it may be distorted into one of the contracted Lie algebras obtained in this article.

A second application that we are actively pursuing is in the theory of special functions. It will be known that multivariable special functions, in particular Appell's generalized hypergeometric functions, can be associated with representations of Lie groups, or quantum groups [30–35]. In most cases the considerations were restricted to polynomials and the relevant group G is compact, in particular $SU(3)$ [31, 33]. If more general special functions are allowed [30], the underlying group will be a non-compact form of the group G, e.g. $SU(2,1)$, $SL(3,\mathbb{R})$, or $SL(3,\mathbb{C})$. Many, possibly all, properties of these special functions then follow from group theoretical considerations. The contractions studied in this article, will then provide relations between the special functions, in particular Appell functions, based on $SL(3,\mathbb{C})$, and those based on the contracted groups.

Realizing the programme of applications outlined above is no mean task. While work in this direction is in progress, reporting on it is well beyond the scope of the present article. However, methods developed in a previous article [25], devoted to contractions of $O(3)$ and the corresponding special function theory are directly applicable to $SL(3,\mathbb{C})$ contrations.

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Appendix. Bases for Lie algebras used in the text and in table 2

For all Lie algebras of dimension $\dim L \leqslant 5$ and for nil-potent Lie algebras of dimension $\dim L \leqslant 6$ we use the notations of [19]. For others we use notations compatible with [6]. Below we give all nonzero commutation relations in a basis $\{x_1, \ldots, x_n\}$:

$A_{2,1}$	$[x_1, x_2] = x_1$			
$A_{3,1}$	$[x_2, x_3] = x_1$			
$A_{3,5}(a)$	$[x_1, x_3] = x_1$	$[x_2, x_3] = ax_2$		
$A_{5,1}$	$[x_3, x_5] = x_1$	$[x_4, x_5] = x_2$		
$A_{5,7}(a, b, c)$	$[x_1, x_5] = x_1$	$[x_2, x_5] = ax_2$	$[x_3, x_5] = bx_3$	$[x_4, x_5] = cx_4$
$A_{5,33}(a, b)$	$[x_1, x_4] = x_1$	$[x_3, x_4] = bx_3$	$[x_2, x_5] = x_2$	$[x_3, x_5] = ax_3$
$A_{5,38}$	$[x_1, x_4] = x_1$	$[x_2, x_5] = x_2$	$[x_4, x_5] = x_3$	
$A_{6,3}$	$[x_1, x_2] = x_6$	$[x_1, x_3] = x_4$	$[x_2, x_3] = x_5$.	

The algebras $A(6)$, $B(6)$, $A(7)$, $B(7)$ and $C(7)$ are all solvable. For $A(6)$ and $A(7)$ the nil-radicals are Abelian, for $B(6)$ the nil-radical is $A_{3,1} \oplus A_1$, for $B(7)$ it is $A_{3,1} + 2A_1$. The nil-radical of $C(7)$ is the non-decomposable nil-potent Lie algebra $A_{5,1}$. In all cases we denote elements of the nil-radical x_i. The two elements in $L/NR(L)$ will be denoted $\{h_1, h_2\}$ and this reflects their origin in the contraction. We shall give the non-zero commutation relations in the nil-radical and represent the action of h_1, h_2 on the nil-radical by two diagonal matrices:

$$[h_1, x_i] = A_{ii}x_i, [h_2x_i] = B_{ii}x_i \tag{A.1}$$

(no summation over i). We have:

$$A(6) \qquad A = \operatorname{diag}(2, -\varepsilon_{02}, -2\varepsilon_{0-1}, \varepsilon_{0-3})$$

$$B = \operatorname{diag}(-1, 2\varepsilon_{02}, \varepsilon_{0-1}, \varepsilon_{0-3})$$

$$A(7) \qquad A = \operatorname{diag}(2, -\varepsilon_{02}, -2\varepsilon_{0-1}, \varepsilon_{0-2}, \varepsilon_{0-3})$$

$$B = \mathrm{diag}(-1, 2\varepsilon_{02}, \varepsilon_{0-1}, -2\varepsilon_{0-2}, \varepsilon_{0-3})$$

$B(6)$

$$A = \mathrm{diag}(2, -1, -2\varepsilon_{0-1}, 1)$$
$$B = \mathrm{diag}(-1, 2, \varepsilon_{0-1}, 1)$$
$$[x_2, x_3] = x_1$$

$B(7)$

$$A = \mathrm{diag}(2, -1, -2\varepsilon_{0-1}, \varepsilon_{0-2}, 1)$$
$$B = \mathrm{diag}(-1, 2, \varepsilon_{0-1}, -2\varepsilon_{0-2}, 1)$$
$$[x_2, x_3] = x_1$$

$C(7)$

$$A = \mathrm{diag}(2, -1, -1, 1, 1)$$
$$B = \mathrm{diag}(-1, 2, -1, -2, 1)$$
$$[x_1, x_2] = x_3, [x_1, x_4] = -x_5$$

(we have $\varepsilon_{0i} \neq 0$ in all cases).

Finally, the algebra $D(7)$ is nil-potent. Its upper central series, lower central series and the derived series are:

$$\mathrm{US} = (3, 7) \qquad \mathrm{CS} = (7, 3, 0) \qquad \mathrm{DS} = (7, 3, 0)$$

and non-zero commutation relations are

$$[x_1, x_2] = x_3 \qquad [x_1, x_4] = x_5 \qquad [x_2, x_6] = x_7.$$

References

[1] Inonu E and Wigner E P 1953 *Proc. Natl. Acad. Sci US* **39** 510
[2] Saletan E J 1961 *J. Math. Phys.* **2** 1
[3] Gilmore R 1974 *Lie Groups, Lie Algebras and Some of their Applications* (New York: Wiley)
[4] Bacry M and Lévy-Leblond J M 1968 *J. Math. Phys.* **9** 1605
[5] Levy-Nahas M 1967 *J. Math. Phys.* **8** 1211
[6] Couture M, Patera J, Sharp R T and Winternitz P 1991 *J. Math. Phys.* **32** 2310–18
[7] Moody R V and Patera J 1991 *J. Phys. A: Math. Gen.* **24** 2227–58
[8] de Montigny M and Patera J 1991 *J. Phys. A: Math. Gen.* **24** 525–47
[9] Leng X and Patera J 1994 *J. Phys. A: Math. Gen.* **27** 1233–50; 1995 *J. Phys. A: Math. Gen.* **28** 3785
[10] de Montigny M, Patera J and Tolar J 1994 *J. Math. Phys.* **35** 405
[11] Weimar-Woods E 1995 *J. Math. Phys.* **36** 4519
[12] Patera J and Zassenhaus H 1989 *Lin. Algebra Applic.* **112** 87–159
[13] Moody R V and Patera J 1984 *J. Math. Phys.* **25** 2838–47
[14] Michel L, Patera J and Sharp R T 1988 *J. Math. Phys.* **29** 777–96
[15] Moody R V and Patera J 1984 *SIAM J. Alg. Discrete Meth.* **5** 359
[16] Rand D, Winternitz P and Zassenhaus H 1988 *Lin. Alg. Appl.* **109** 197–246
[17] Jacobson N 1979 *Lie Algebras* (New York: Dover)
[18] Zassenhaus H 1981 *Lie Groups, Lie Algebras and Representation Theory* (Montréal: Les Presses de l'Université de Montréal)
[19] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 *J. Math. Phys.* **17** 986
[20] Kirillov A A 1976 *Elements of the Theory of Representations* (New York: Springer)
[21] Abellanas L and Martinez Alonso L 1975 *J. Math. Phys.* **16** 1580
[22] Perroud M 1983 *J. Math. Phys.* **24** 1381
[23] Ndogmo J C and Winternitz P 1994 *J. Phys. A: Math. Gen.* **27** 405; 1994 *J. Phys. A: Math. Gen.* **27** 2787
[24] Rubin J and Winternitz P 1993 *J. Physique* A **26** 1123
[25] Izmest'ev A A, Pogosyan G S, Sissakian A N and Winternitz P 1996 *J. Phys. A: Math. Gen.* **29** 1
[26] Kadyshevsky V G 1962 *Dokl. Akad. Nauk SSSR* **147** 588
[27] de Montigny M 1993 *Nuovo Cimento* B **108** 1171
[28] Ait Abdelmalek M 1995 *PhD Thesis* Université de Montréal

[29] Nono T 1968 *J. Math. Anal. Appl.* **24** 110
[30] Charap J M and Minton B M 1969 *J. Math. Phys.* **10** 1823
[31] Balachandran A P and Nuyts J 1968 *Phys. Rev.* **172** 1821
[32] Balachandran A P, Meggs W J, Nuyts J and Ramond P 1968 *Phys. Rev.* **176** 1700
[33] Beg M A B and Ruegg H 1965 *J. Math. Phys.* **6** 677
[34] Floreanini R, Lapointe L and Vinet L 1994 *J. Physique* A **27** 6781
[35] Lapointe L and Vinet L 1993 *Appell Functions and the Pöschl-Teller Potential* CRM-1925 Montréal